Chapter 3
FUZZY LOGIC

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3.1 Introduction

➢ **Fuzzy logic is a branch of logic designed for representing knowledge and human reasoning in such a way that it is possible to process by a computer.**

➢ **The traditional propositional and predicate logic do not allow for degrees of imprecision, indicated by words or phrases such as fairly, very and quite possibly.**

➢ **Instead of truth values such as true and false, it is possible to introduce a multi-valued logic, like**

- ❖ true
- ❖ not true
- ❖ very true
- ❖ not very true
- ❖ not very false
- ❖ very false
- ❖ not false
- ❖ false

3.1 Introduction

➢ Fuzzy systems implement fuzzy logic, which uses sets and predicates of this kind.

➢ The main concepts of fuzzy logic are

- ❖ fuzzy sets
- ❖ linguistic variables
- ❖ possibility distributions
- ❖ fuzzy IF-THEN rules

3.2 Foundation of Fuzzy Systems

➢ Fuzziness pertains to the uncertainty associated with a system, i.e., the fact that nothing can be predicated with exact precision.

➢ All real life situations have some degree of uncertainty or fuzziness.

➢ In 1965 Lotfi A. Zadeh introduced fuzzy sets, with which a more flexible sense of membership is possible.

➢ However, practically the value of variables is not always known precisely.

➢ So, there is some uncertainty or vagueness associated with system variables.

3.2 Foundation of Fuzzy Systems

- One cannot adequately express this uncertainty using a '*crisp*' variable.
- The vagueness can adequately be handled using fuzzy set theory.
- This theory provides a strict mathematical framework using which vague conceptual phenomena can be studied.
- In this theory, the variables, functions, etc. connected with the imprecise phenomenon to be studied are expressed as *fuzzy variables* and *fuzzy functions*.

3.3 Fuzzy Vs Crisp

- A crisp set is a collection of distinct (precisely defined) elements.
- In classical set theory, a crisp set can be a superset containing other crisp sets.
- Consider the query, "Is water colorless?"
- The answer to this is a definite Yes/True or No/False, as warranted by the situation.
- If Yes/True is accorded the value of 1 and No/False is 0, this statement results in a 0/1 type of situation.
- Such logic which demands a binary (0/1) type of handling is termed *crisp* in the domain of fuzzy set theory.

3.3 Fuzzy Vs Crisp

- Some examples of crisp situations are
  - Temperature is 32° C.
  - The running time of program is 4 seconds.
- On the other hand, consider the statement, "Is a leader honest?"
- The answer to this query need not be a definite "yes" or "no".
- Considering the degree to which one knows a leader, a variety of answers can be generated.
  - Extremely honest
  - Extremely dishonest
  - Honest at times
  - Very honest

3.3 Fuzzy Vs Crisp

- If, for instance,
  - "extremely honest" were to be assigned a value of 1, at the high end of the spectrum of values
  - "extremely dishonest" were to be assigned a value of 0, at the low end of the spectrum of values
  - "honest at times" could be assigned value of 0.4
  - "very honest" could be assigned value of 0.85
- The situation is so fluid that it can accept values between 0 and 1, in contrast to the earlier one which was either a 0 or 1.
- Such a situation is called *fuzzy*.
- Classical set (crisp set) theory is fundamental to the study of fuzzy sets.

3.3.1 Crisp Sets

Universe of discourse

- **The universe of discourse or universal set is the set which, with reference to a particular context, contains all possible elements having the same characteristics and from which sets can be formed.**
- **It is denoted by U or E.**
- **Example,**
  1) **The universal set of all numbers in Euclidean space.**
  2) **The universal set of all students in a university.**

3.3.1 Crisp Sets

Set

- **A set is well defined collection of objects.**
- **The object either belongs to or does not belong to the set.**
- **Consider a set A whose objects are $a_1, a_2, ..., a_n$.**
- **Here $a_1, a_2, ..., a_n$ are called the *members* of the set.**
- **Example,**
  1) **A = {Cricket, Football, Tennis}**
  2) **B = {Swan, Peacock, Dove}**
  3) **C = {GCET, ADIT, BVM}**

3.3.1 Crisp Sets

Set

- A **set** may also be defined based on the properties the members have to satisfy.
- In such a case, a set **A** is defined as

$$A = \{x \mid P(x)\}$$

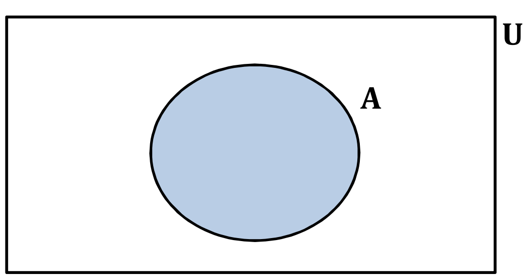
- Here, **P(x)** stands for the property **P** to be satisfied by the member **x**.
- This is read as '**A is the set of all X such that P(x) is satisfied**'.
- Example,

1) $A = \{x \mid x \text{ is an odd number}\}$
2) $B = \{y \mid y > 0 \text{ and } y \bmod 5 = 0\}$

3.3.1 Crisp Sets

Venn diagram

- **Venn diagrams** are pictorial representations to denote a set.
- Given a set **A** defined over a universal set **U**, the Venn diagram for **A** and **U** is as shown in figure below.



3.3.1 Crisp Sets

Membership

➢ An element x is said to be a member of a set A if x belongs to the set A.

➢ The membership is indicated by $\in$ and is pronounced "belongs to".

$x \in A$ -- Means x belongs to A

$x \notin A$ -- Means x does not belong to A

Cardinality

➢ The number of elements in a set is called its cardinality.

➢ Cardinality of a set A is denoted as n(A) or |A| or #A.

➢ Example

If A = {4,5,6,7} then |A| = 4

3.3.1 Crisp Sets

Family of sets

➢ A set whose members are sets themselves, is referred to as a family of sets.

Example

A = {{4,5,6,7}, {1,2,3}, {6,12}} is a set whose members are the sets {4,5,6,7}, {1,2,3} and {6,12}

Null Set/Empty Set

➢ A set is said to be a null or empty set if it has no members.

➢ A null set is indicated as Φ or { } and indicates an impossible event.

➢ Also, |Φ| = 0.

3.3.1 Crisp Sets

Subset

- Given sets **A** and **B** on **U** the universal set, **A** is said to be a subset of **B** if **A** is fully contained in **B**, that is, every element of **A** is in **B**.
- It is denoted by $A \subset B$ we say that **A** is a **subset** of **B**, or **A** is a ***proper subset*** of **B**.
- If **A** is contained in or equivalent to that of **B** then we denote the subset relation as $A \subseteq B$
- In such a case, **A** is called the ***improper subset*** of **B**.

Superset

- Given sets **A** and **B** on **U** the universal set, **A** is said to be a **superset** of **B** if every element of **B** is contained in **A**.
- It is denoted by $A \supset B$ we say that **A** is a superset of **B**, or **A** is a ***proper subset*** of **B**.

3.3.1 Crisp Sets

Superset

- If **A** contains **B** and is equivalent to that of **B** then we denote the subset relation as $A \supseteq B$

Power set

- A **power set** of a set **A** is the set of all possible subsets that are derivable from **A** including null set.
- A power set is indicated as **P(A)** and has cardinality of $|P(A)| = 2^{|A|}$

Example:

$$A = \{3, 4, 6\}$$

$$P(A) = \{\{3\}, \{4\}, \{6\}, \{3, 4\}\{3, 6\}, \{4, 6\}\{3, 4, 6\}, \text{¤}\}$$

Here, **|A| = 4** and $|P(A)| = 2^3 = 8$

3.3.2 Operations on Crisp Sets

Union (U)

➢ The **union** of two sets **A** and **B** (**A U B**) is the set of all elements that belong to **A** or **B** or **both**.

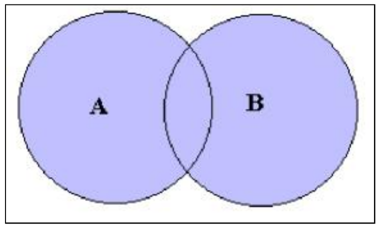
$$A \cup B = \{x \mid x \in A \text{ or } x \in \mathrm{B}\}$$

Example:

$$A = \{a, b, c, 1, 2\}, \quad B = \{1, 2, 3, a, c\}$$

➢ So, we get, $A \cup B = \{a, b, c, 1, 2, 3\}$

➢ The Venn diagram of **A U B** is shown in the figure.



3.3.2 Operations on Crisp Sets

Intersection (∩)

➢ The **intersection** of two sets **A** and **B** (**A ∩ B**) is the set of all elements that belong to **A** and **B**.

$$A \cap B = \{x \mid x \in A \text{ and } x \in \mathrm{B}\}$$

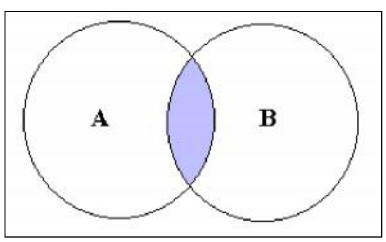
Example:

$$A = \{a, b, c, 1, 2\}, \quad B = \{1, 2, 3, a, c\}$$

➢ So, we get, $A \cap B = \{a, c, 1, 2\}$

➢ The Venn diagram of **A ∩ B** is shown in the figure.

➢ Any two sets **A** and **B**, which have **A ∩ B = Φ** are called **Disjoint** sets.



3.3.2 Operations on Crisp Sets

Complement (^{c})

- **The complement of a set A (Ā/A^c) is the set of all elements which are in U but not in A.**

$$A^c = \{x \mid x \notin A, x \in U\}$$

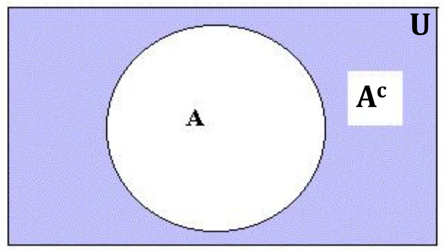
Example:

- **Given**

$$U = \{1, 2, 3, 4, 5, 6, 7\} \text{ and } A = \{3, 4, 5\}$$

- **We get** $A^c = \{1, 2, 6, 7\}$

- **The Venn diagram of A^c is shown in the figure.**



3.3.2 Operations on Crisp Sets

Difference (-)

- **The difference of two sets A and B (A - B) is the set of all elements which are in A but not in B.**

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

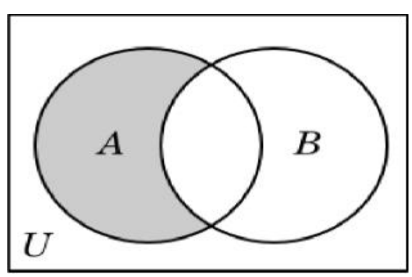
Example:

$$A = \{a, b, c, d, e\}, \quad B = \{b, d\}$$

- **So, we get,**

$$A - B = \{a, c, e\}$$

- **The Venn diagram of A - B is shown in the figure.**



3.3.3 Properties of Crisp Sets

- **Following are the properties of sets.**

- **Commutativity:**

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- **Associativity:**

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- **Distributivity:**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **Idempotence:**

$$A \cup A = A$$
$$A \cap A = A$$

3.3.3 Properties of Crisp Sets

- **Identity:**

$$A \cup ¤ = A$$
$$A \cap U = A$$
$$A \cap ¤ = ¤$$
$$A \cup U = U$$

- **Law of Absorption:**

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

- **Transitivity:**

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

3.3.3 Properties of Crisp Sets

- **Involution:** $$(A^c)^c = A$$

- **Law of the Excluded Middle:** $$A \cup A^c = U$$

- **Law of Contradiction:** $$A \cap A^c = \emptyset$$

- **De Morgan's Laws:**
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

3.4 Fuzzy sets

- **In classical set theory, an element either belongs to a set or does not belong to it.**

- **If the set under consideration is A, the testing of an element x using the characteristic function $\mu_A(x)$ is expressed as**

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

- **In fuzzy sets many degrees of membership are allowed.**

- **A number between 0 and 1 indicates the degree of membership to a set in the interval [0,1].**

- **The point of departure for fuzzy sets is the generalization of the valuation set from {0,1} to all numbers found in the interval [0,1].**

3.4 Fuzzy sets

- By expanding the valuation set, we alter the nature of the characteristic function which is called ***membership function***, denoted by **$\mu_A(x)$**.

- **The membership function can take intermediate values between 1 and 0 and is indicated by square bracket [0,1].**

- **Since the interval [0,1] contains infinite members, infinite degrees of membership are possible.**

$$\mu_{\tilde{A}}(x) : X \rightarrow [0,1] \ldots\ldots(3.1)$$

- **There are two commonly used ways of denoting fuzzy sets.**

- **If X is the universe of discourse and x is a particular element of X, then a fuzzy set A defined on X may be written as**

3.4 Fuzzy sets

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X\} \ldots\ldots(3.2)$$

- **In above equation 3.2, each pair $(x, \mu_A(x))$ is called a *singleton* and shows x followed by its membership in A, $\mu_A(x)$.**

- **In crisp sets, a singleton is the element x by itself.**

- **In fuzzy sets, a singleton is composed of two terms: x and $\mu_A(x)$.**

- **A singleton is also written as $\mu_A(x) / x$.**

- **In above representation marker '/' is not division but is used to separate the function from x.**

3.4 Fuzzy sets

➢ For example, a set of small integers **A** over the universe of discourse of positive integers may be given by the collection of all singletons $\mu_A(x_i) / x_i$ as

$$A = \{(1,1.0),(2,1.0),(3,0.75),(4,0.5),(5,0.3),(6,0.3),(7,0.1),(8,0.1)\} \quad ....(3.3)$$

➢ The fuzzy set **A** can also be represented as

$$A = \sum_{x_i \in X} \mu_{\tilde{A}}(x_i) / x_i \quad ....(3.4)$$

➢ An alternative notation used to represent a fuzzy set as the union of all $\mu_A(x_i) / x_i$ singletons:

$$A = \bigcup_{x_i \in X} \mu_{\tilde{A}}(x_i) / x_i$$

3.4 Fuzzy sets

➢ In this alternative notation, the set of small integers mentioned above may be written as

$$\begin{aligned} A &= \mu_A(1)/1 + \mu_A(2)/2 + \mu_A(3)/3 + \mu_A(4)/4 + \\ &\quad \mu_A(5)/5 + \mu_A(6)/6 + \mu_A(7)/7 + \mu_A(8)/8 \\ &= 1.0/1+1.0/2+0.75/3+0.5/4+0.3/5+0.3/6+0.1/7+0.1/8 \end{aligned}$$

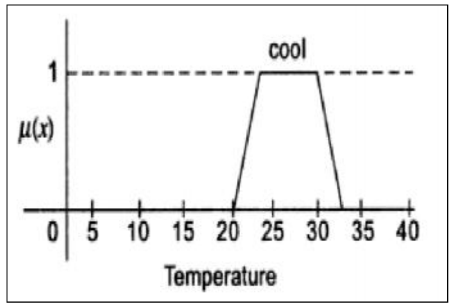
➢ For a continuous universe of discourse, we write

$$A = \int_X \mu_{\tilde{A}}(x) / x \quad ....(3.5)$$

➢ Where the integer sign represents the union of all $\mu_A(x) / x$ singletons.

3.4 Fuzzy sets
3.4.1 Membership Function

- **The membership function values need not always to be described by values.**
- **Quite often, these turn out to be as described by a continuous function.**
- **The fuzzy membership function for the fuzzy linguistic term "cool" relating to temperature may turn out to be as illustrated in following figure.**

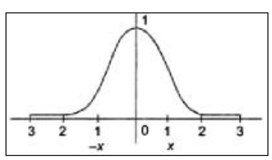


- **The membership function is mathematically given by**

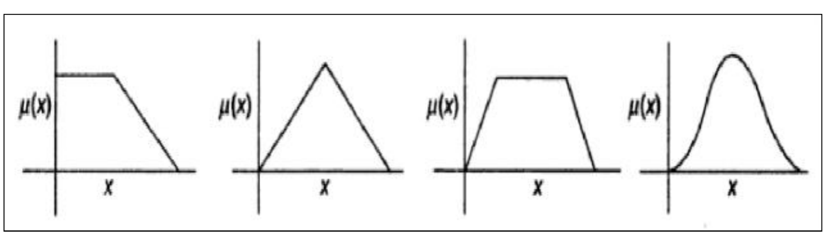
$$\mu_{\tilde{A}}(x) = \frac{1}{(1+x)^2}$$

3.4 Fuzzy sets
3.4.1 Membership Function

- **The graph is shown in following figure.**



- **Different shapes of membership functions exist like triangular, trapezoidal, curved, etc. as shown below.**



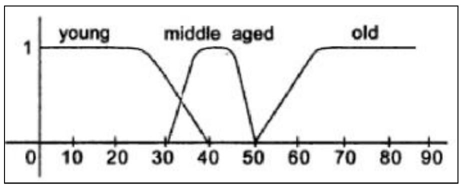
3.4 Fuzzy sets
3.4.1 Membership Function

Example:

- **Consider the set of people in the following age groups**

0-10	40-50
10-20	50-60
20-30	60-70
30-40	70 and above

- **The fuzzy sets "young", "middle-aged" and "old" are represented by membership function graph as**



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3.4.2 Basic Terms and Operations

- **Some of the operations such as intersection and union are defined through the min (∧) and max (∨) operators.**

- **These are analogous to the product (•) and sum (+) in algebra.**

- **Min (∧) and max (∨) may be used to select the minimum and maximum of two elements respectively.**

- **For example,**

$$3 \wedge 4 = 3$$

$$3 \vee 4 = 4$$

- **These can also be expressed as**

$$\min(3,4) = 3$$

$$\max(3,4) = 4$$

3.4.2 Basic Terms and Operations

➢ In general, the minimum of two elements μ_1 and μ_2 can be denoted by one of the three methods given

$$\min(\mu_1, \mu_2) \quad \text{(i)}$$
$$\wedge(\mu_1, \mu_2) \quad \text{(ii)}$$
$$\mu_1 \wedge \mu_2 \quad \text{(iii)}$$

➢ It is defined as

$$\mu_1 \wedge \mu_2 = \begin{cases} \mu_1 \text{ iff } \mu_1 \leq \mu_2 \\ \mu_2 \text{ iff } \mu_2 \leq \mu_1 \end{cases} \quad ....(3.6)$$

➢ where "iff" is a shorthand for "if and only if".

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3.4.2 Basic Terms and Operations

➢ Similarly, the maximum of two elements μ_1 and μ_2 can be denoted as $\mu_1 \vee \mu_2$

$$\mu_1 \vee \mu_2 = \begin{cases} \mu_1 \text{ iff } \mu_1 \geq \mu_2 \\ \mu_2 \text{ iff } \mu_2 \geq \mu_1 \end{cases} \quad ....(3.7)$$

➢ **Min** and **max** can operate on an entire set, selecting the smallest element (***infimum***) or the largest element (***supremum***) of the set.

Empty Fuzzy Set

➢ A set is said to be empty if it has no members and its membership function is zero everywhere in its universe of discourse X.

$$A \equiv \emptyset \text{ if } \mu_A(x)=0 \quad \forall x \in X \quad ....(3.8)$$

3.4.2 Basic Terms and Operations

Normal Fuzzy Set

- **A normal fuzzy set has a membership function that includes at least one singleton equal to unity in its universe of discourse.**
- **If there is no singleton equal to unity, the fuzzy set is called *subnormal*.**
- **For a normal fuzzy set,**

$$«_A(x)=1 \quad ....(3.9)$$

Equality of Fuzzy Sets

- **Two fuzzy sets are said to be equal if their membership functions are equal everywhere in the universe of discourse.**

$$A \equiv B \text{ if } «_A(x)=«_B(x) \quad ....(3.10)$$

3.4.2 Basic Terms and Operations

Union of two fuzzy set

- **The union of two fuzzy sets A and B defined over the same universe of discourse X is a new fuzzy set.**
- **It is a set with a membership function that represents the maximum degree of relevance between each element and the new fuzzy set.**
- **The membership function of the new fuzzy set is expressed as**

$$«_{A\cup B}(x) \equiv «_A(x) \vee «_B(x) \quad ......(3.11)$$

- **The union of two fuzzy sets is related to the logical operation of *disjunction (OR)* in fuzzy logic.**

3.4.2 Basic Terms and Operations

Intersection of two fuzzy set

➢ **The intersection of two fuzzy sets A and B defined over the same universe of discourse X is a new fuzzy set.**

➢ **It is a set with a membership function that represents the minimum degree of relevance between each element and the new fuzzy set.**

➢ **The membership function of the new fuzzy set is expressed as**

$$\mu_{A \cap B}(x) \equiv \mu_A(x) \wedge \mu_B(x) \quad ......(3.12)$$

➢ **The intersection of two fuzzy sets is related to the logical operation of *conjunction (AND)* in fuzzy logic.**

3.4.2 Basic Terms and Operations

Complement of a fuzzy set

➢ **The complement of a fuzzy set A is a new set Ā containing all the elements which are in the universe of discourse but not in A.**

$$\mu_{\bar{A}}(x) \equiv 1 - \mu_A(x) \quad ......(3.13)$$

Product of two fuzzy sets

➢ **The product of two fuzzy sets A and B defined over the same universe of discourse X is a new fuzzy set.**

➢ **A new fuzzy set with a membership function that equals the *algebraic product* of the membership functions of A and B.**

$$\mu_{A \bullet B}(x) \equiv \mu_A(x) \bullet \mu_B(x) \quad ......(3.14)$$

➢ **The *algebraic product* is the multiplication of the possibility values of each corresponding singleton.**

3.4.2 Basic Terms and Operations

Power of a fuzzy set

- **The α[th] power of a fuzzy set Ā is a new fuzzy set A^α whose membership function is given by**

$$\mu_{A^\alpha}(x) \equiv [\mu_A(x)]^\alpha \quad ......(3.15)$$

- **Raising a fuzzy set to the second power is usually taken to be equivalent to linguistically applying the modifier VERY to it.**

- **The square of the membership function of B = (large numbers) represents the fuzzy set B^2=(Very Large numbers).**

- **Taking the square root of a fuzzy set is called dilation or DIL.**

3.4.2 Basic Terms and Operations

Concentration

- **The concentration of a fuzzy set A defined over a universe of discourse X is denoted as CON(A) and is a new fuzzy set with its membership function is given by**

$$\mu_{CON(A)}(x) \equiv [\mu_A(x)]^2 \quad ......(3.16)$$

Dilation

- **The dilation of a fuzzy set A, denoted as DIL(A), produces a new fuzzy set in X with its membership function defined as the square root of the membership function of A.**

$$\mu_{DIL(A)}(x) \equiv \sqrt{\mu_A(x)} \quad ......(3.17)$$

3.4.2 Basic Terms and Operations

Difference

- **The difference of two fuzzy sets A & B is a new fuzzy set A – B defined as**

$$\tilde{A} - \tilde{B} = (\tilde{A} \cap \tilde{B}^c) \;......(3.18)$$

Disjunctive Sum

- **The disjunctive sum of two fuzzy sets A and B is a new fuzzy set A B defined as**

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c) \;......(3.19)$$

3.4.3 Properties of Fuzzy Sets

Cardinality

- **The cardinality of a set is the total number of elements in the set.**
- **Since an element can partially belongs to a fuzzy set, a natural generalization of the classical notion of cardinality is to weigh each element by its membership degree.**
- **The following formula for calculating the cardinality of a fuzzy set**

$$\text{Card}(A) = \sum_{x_i} \mu_A(x_i)$$

- **The cardinality of a set is used in defining other properties such as the normalization factor.**

3.4.3 Properties of Fuzzy Sets

Height (normal versus subnormal)

➢ **The height of a fuzzy set is the highest membership value of its membership function:**

$$\text{Height(A)} = \max_{x_i} \mu_A(x_i)$$

➢ **For example, let us consider the height of a fuzzy set to be 0.5.**

➢ **A fuzzy set with height 1 is called a normal fuzzy set and a set whose height is less than 1 is called a subnormal fuzzy set.**

➢ **A subnormal fuzzy set is set containing only partial members, no full members.**

➢ **So, it figures between an empty set and a non-empty set in the classical sense.**

3.4.3 Properties of Fuzzy Sets

Convexity

➢ **A fuzzy set is convex if its membership function does not have a 'valley'.**

➢ **Let X be the universe of discourse of a variable x.**

➢ **Let A be a fuzzy subset of X.**

➢ **The set A is convex if and only if**

$$\mu_A(\lambda a + [1-\lambda]b) \geq \min\{\mu_A(a), \mu_A(b)\}$$

$$(\textit{for} \text{ all a,b } \in \text{ X and } 0 \leq \lambda \leq 1)$$

➢ **This condition states that the membership value of any given element in the interval [a,b] should not be less than the membership value of either end point.**

3.4.3 Properties of Fuzzy Sets

- **Commutativity:**

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$$
$$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$$

- **Associativity:**

$$(\tilde{A} \cup \tilde{B}) \cup \tilde{C} = \tilde{A} \cup (\tilde{B} \cup \tilde{C})$$
$$(\tilde{A} \cap \tilde{B}) \cap \tilde{C} = \tilde{A} \cap (\tilde{B} \cap \tilde{C})$$

- **Distributivity:**

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$
$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$$

- **Idempotence:**

$$\tilde{A} \cup \tilde{A} = \tilde{A}$$
$$\tilde{A} \cap \tilde{A} = \tilde{A}$$

3.4.3 Properties of Fuzzy Sets

- **Identity:**

$$\tilde{A} \cup \emptyset = \tilde{A}$$
$$\tilde{A} \cap X = \tilde{A}$$
$$\tilde{A} \cap \emptyset = \emptyset$$
$$\tilde{A} \cup X = X$$

- **Law of Absorption:**

$$\tilde{A} \cup (\tilde{A} \cap \tilde{B}) = \tilde{A}$$
$$\tilde{A} \cap (\tilde{A} \cup \tilde{B}) = \tilde{A}$$

- **Transitivity:**

$$\textit{If } \tilde{A} \subseteq \tilde{B}, \tilde{B} \subseteq \tilde{C} \text{ then } \tilde{A} \subseteq \tilde{C}$$

3.4.3 Properties of Fuzzy Sets

- **Involution:** $(\tilde{A}^c)^c = \tilde{A}$

- **Law of the Excluded Middle:** $\tilde{A} \cup \tilde{A}^c \neq X$

- **Law of Contradiction:** $\tilde{A} \cap \tilde{A}^c \neq$ ¤

- **De Morgan's Laws:**

$$(\tilde{A} \cup \tilde{B})^c = \tilde{A}^c \cap \tilde{B}^c$$
$$(\tilde{A} \cap \tilde{B})^c = \tilde{A}^c \cup \tilde{B}^c$$

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3.5 Fuzzy Relations

- **Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets X_1, X_2,....X_n where the n-tuples (x_1,x_2,...,x_n) may have varying degrees of membership within the relation.**
- **The membership values indicate the strength of the relation between the inputs.**

Example:

- **Let R be the fuzzy relation between two sets X_1 and X_2 where X_1 is the set of diseases and X_2 is the set of symptoms.**

X_1 = {typhoid, viral fever, common cold}

X_2 = {running nose, high temperature, shivering}

	Running Nose	High Temp.	Shivering
Typhoid	0.1	0.9	0.8
Viral Fever	0.2	0.9	0.7
Common Cold	0.9	0.4	0.6

3.5.1 Fuzzy Cartesian Product

- Let **A** be a fuzzy set defined on the universe **X** and **B** be a set defined on the universe **Y**, the Cartesian product between the fuzzy set **A** and **B** indicated as **A x B** and resulting in a fuzzy relation **R** is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y ....(3.20)$$

where **R** has its membership function given by

$$\mu_R(x, y) = \mu_{A \times B}(x, y)$$
$$= \min(\mu_A(x), \mu_B(y)).....(3.21)$$

3.5.2 Operations on Fuzzy Relations

- Let **R** and **S** be fuzzy relations on **X x Y**

Union

$$\mu_{R \cup S}(x, y) = \max(\mu_R(x, y), \mu_S(x, y)).....(3.22)$$

Intersection

$$\mu_{R \cap S}(x, y) = \min(\mu_R(x, y), \mu_S(x, y)).....(3.23)$$

Complement

$$\mu_{R^c}(x, y) = 1 - \mu_R(x, y).....(3.24)$$

3.5.2 Operations on Fuzzy Relations

Composition of Relations

- **Let R and S be fuzzy relations on X x Y, and S is a fuzzy relation defined on Y x Z, then R ∘S is a fuzzy relation on X x Z.**

- **The fuzzy max-min composition is defined as**

$$\mu_{R \circ S}(x,z) = \max_{y \in Y}(\min(\mu_R(x,y), \mu_S(y,z))).....(3.25)$$

3.5.3 Fuzzy Propositions

- **Fuzzy proposition is a statement which acquires a fuzzy truth value.**

- **Thus, given P to be a fuzzy proposition, T(P) represents the truth value (0-1) attached to P.**

- **In its simplest form, fuzzy propositions are associated with fuzzy sets.**

- **The fuzzy membership value associated with the fuzzy set A for P is treated as the fuzzy truth value T(P).**

$$T(P) = \mu_A(x) \text{ where } 0 \le \mu_A(x) \le 1.....(3.26)$$

3.5.4 Fuzzy Connectives

- Fuzzy logic similar to crisp logic supports the following connectives:

(i) Negation: -
(ii) Disjunction: $\vee$
(iii) Conjuction: $\wedge$
(iv) Implication: $\Rightarrow$

- Following table illustrates the connectives for fuzzy propositions **P** and **Q**.

Symbol	Connective	Usage	Definition
-	Negation	$\bar{P}$	1-T(P)
∨	Disjunction	$P \vee Q$	max(T(P),T(Q))
∧	Conjunction	$P \wedge Q$	min(T(P),T(Q))
▯	Implication	$P \Rightarrow Q$	$\bar{P} \vee Q = \max(1-T(P), T(Q))$

3.5.4 Fuzzy Connectives

- **P** and **Q** related by the '▯' operator are called **antecedent** and **consequent** respectively.
- The '▯' operator represents the **IF-THEN** statement as

IF x is A THEN y is B

$$R = (A \times B) \cup (\bar{A} \times Y)......(3.27)$$

- The membership function of **R** is given by

$$«_R(x, y) = \max(\min(«_A(x), «_B(y)), 1 - «_A(x)).....(3.28)$$

- Also for compound implication **IF x is A THEN y is B ELSE y is C**, the relation is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)......(3.29)$$

$$«_R(x, y) = \max(\min(«_A(x), «_B(y)), \min(1 - «_A(x), «_{\bar{C}}(y))...(3.30)$$

3.5.4 Fuzzy Inference

- Fuzzy inference also known as *approximate reasoning* refers to computational procedures used for evaluating linguistic description.
- The two important inferring procedures are
  1) Generalized Modus Ponens (GMP)
  2) Generalized Modus Tollens (GMT)
- Let us consider a linguistic description involving only a simple IF-THEN rule with a known implication relation R(x,y) and a fuzzy value A' approximately matching the antecedent of the rule.
- GMP allows us to compute or infer the consequent B'.
- Similarly, using GMT, A' can be inferred if we know B'.

3.5.4 Fuzzy Inference

- For example, 'If the question paper is EASY then the performance is VERY GOOD'.
- Given that 'question paper is AVERAGE', GMP allows us to evaluate the rule and infer a value for the performance.
- GMP is formally stated as

IF x is A THEN y is B

x is A'

---

y is B'

- To compute the membership function of B', we use the max-min composition of the fuzzy set A' with R(x,y).

$$B' = A' \circ R(x, y) \quad ......(3.31)$$

3.5.4 Fuzzy Inference

- In terms of membership function,

$$\mu_{B'}(y) = \max(\min(\mu_{A'}(x), \mu_R(x, y))) \quad ......(3.32)$$

- **GMT** has the form

IF x is A THEN y is B

y is B'

---

x is A'

- The membership of A' is computed as,

$$A' = B' \circ R(x, y) \quad ......(3.33)$$

- In terms of membership function,

$$\mu_{A'}(y) = \max(\min(\mu_{B'}(y), \mu_R(x, y))) \quad ......(3.34)$$

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3.5.5 Fuzzy Inference: Mamdani

- The most commonly used fuzzy inference technique is the so-called **Mamdani** method.
- In 1975, **Prof. Ebrahim Mamdani** of London University built one of the first fuzzy systems to control a steam engine and boiler combination.
- The **Mamdani-style** fuzzy inference process is performed in four steps:

  1) Fuzzification of the input variables,

  2) Rule evaluation,

  3) Aggregation of the rule outputs, and finally

  4) Defuzzification.

3.5.5 Fuzzy Inference: Mamdani

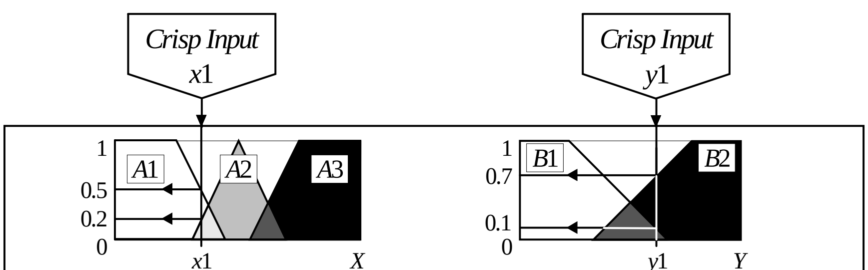
➢ We examine a simple two-input one-output problem that includes three rules:

Rule: 1	**Rule: 1**
IF x is A3	IF project_funding is adequate
OR y is B1	OR project_staffing is small
THEN z is C1	THEN risk is low
Rule: 2	**Rule: 2**
IF x is A2	IF project_funding is marginal
AND y is B2	AND project_staffing is large
THEN z is C2	THEN risk is normal
Rule: 3	**Rule: 3**
IF x is A1	IF project_funding is inadequate
THEN z is C3	THEN risk is high

3.5.5 Fuzzy Inference: Mamdani

Step 1: Fuzzification

➢ The first step is to take the crisp inputs, x1 and y1 (project funding and project staffing), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



$\mu_{(x=A1)} = 0.5$
$\mu_{(x=A2)} = 0.2$

$\mu_{(y=B1)} = 0.1$
$\mu_{(y=B2)} = 0.7$

3.5.5 Fuzzy Inference: Mamdani

Step 2: Rule Evaluation

- **The second step is to take the fuzzified inputs, $\text{«}_{(x=A1)}$ = 0.5, $\text{«}_{(x=A2)}$ = 0.2, $\text{«}_{(y=B1)}$ = 0.1 and $\text{«}_{(y=B2)}$ = 0.7, and apply them to the antecedents of the fuzzy rules.**

- **If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.**

- **This number (the truth value) is then applied to the consequent membership function.**

3.5.5 Fuzzy Inference: Mamdani

Step 2: Rule Evaluation

- **To evaluate the disjunction of the rule antecedents, we use the OR fuzzy operation.**

- **Typically, fuzzy expert systems make use of the classical fuzzy operation union:**

$$\text{«}_{A \text{Ć} B}(x) = max\ [\text{«}_A(x), \text{«}_B(x)]$$

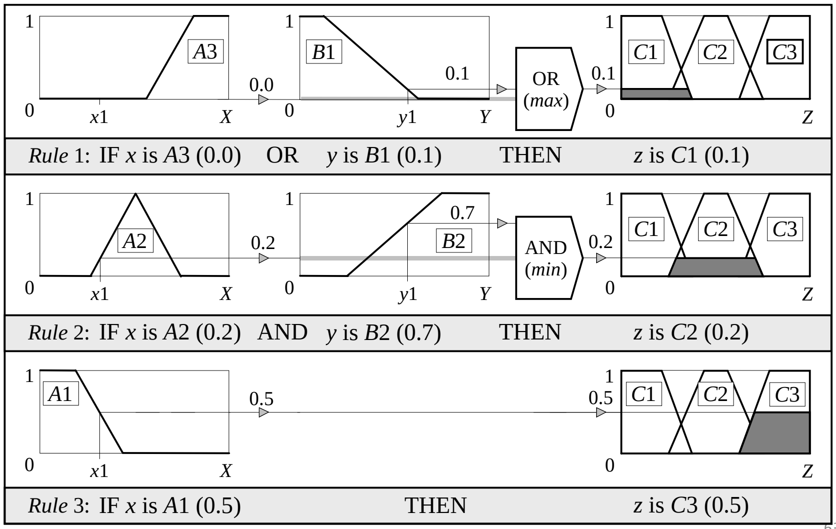
- **Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the AND fuzzy operation intersection:**

$$\text{«A ą B(x) = min [«A(x), «B(x)]}$$

3.5.5 Fuzzy Inference: Mamdani

Step 2: Rule Evaluation

Mamdani-style rule evaluation



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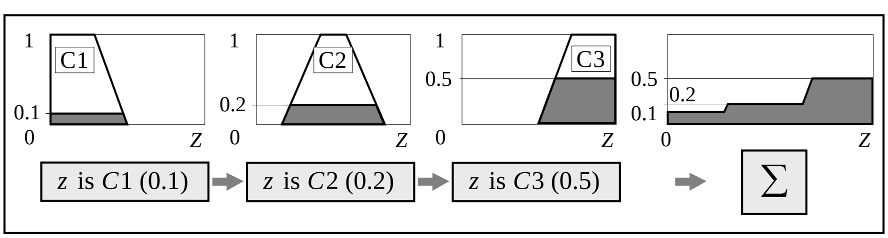
3.5.5 Fuzzy Inference: Mamdani

Step 3: Aggregation of the rule outputs

- **Aggregation is the process of unification of the outputs of all rules.**
- **We take the membership functions of all rule consequents and combine them into a single fuzzy set.**
- **The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.**



3.5.5 Fuzzy Inference: Mamdani

Step 4: Defuzzification

- **The last step in the fuzzy inference process is defuzzification.**
- **Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.**
- **The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.**
- **Following defuzzification techniques are used frequently**
  - **Centre of Area**
  - **Centre of Sum**
  - **Mean of Maxima**

3.5.6 Fuzzy Inference: Sugeno

- **In Mamdani-style inference, it requires to find the centroid of a two-dimensional shape by integrating across a continuously varying function.**
- **In general, this process is not computationally efficient.**
- **Michio Sugeno suggested to use a single spike, a *singleton*, as the membership function of the rule consequent.**
- **A singleton, or more precisely a fuzzy singleton, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.**
- **Sugeno-style fuzzy inference is very similar to the Mamdani method.**

3.5.6 Fuzzy Inference: Sugeno

➢ Sugeno changed only a rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable.

➢ The format of the Sugeno-style fuzzy rule is

IF x is A

AND y is B

THEN z is f(x, y)

Where

❖ x, y and z are linguistic variables;

❖ A and B are fuzzy sets on universe of discourses X and Y, respectively;

❖ f (x, y) is a mathematical function.

3.5.6 Fuzzy Inference: Sugeno

➢ The most commonly used zero-order Sugeno fuzzy model applies fuzzy rules in the following form:

IF x is A

AND y is B

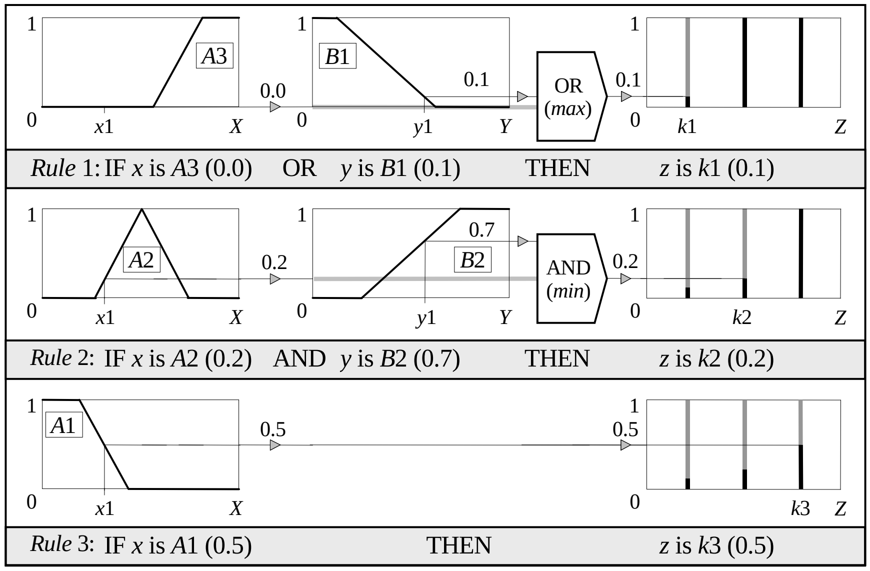
THEN z is k

➢ where k is a constant.

➢ In this case, the output of each fuzzy rule is constant.

➢ All consequent membership functions are represented by singleton spikes.

3.5.6 Fuzzy Inference: Sugeno



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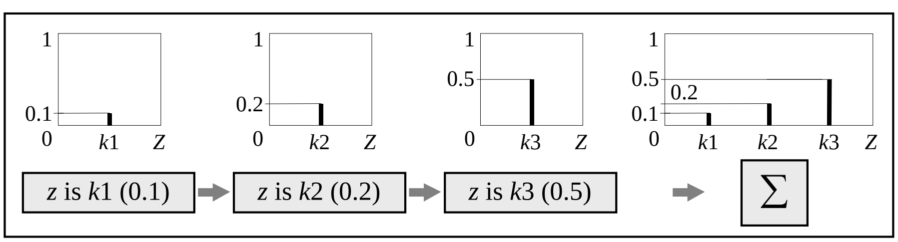
3.5.6 Fuzzy Inference: Sugeno

Sugeno-style aggregation of the rule outputs



3.5.6 Fuzzy Inference

- How to make a decision on which method to apply J Mamdani or Sugeno?
- Mamdani method is widely accepted for capturing expert knowledge.
- It allows us to describe the expertise in more intuitive, more human-like manner.
- However, Mamdani-type fuzzy inference entails a substantial computational burden.
- Sugeno method is computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.

3.6 Fuzzy Rule Based System

- The fuzzy implication $P \rightarrow Q$ is a mechanism for GMP inference applicable to a chain of conditional propositions.
- In unconditional reasoning, the relationship of interest is the association between the composed elements.
- In the conditional case, the first event constitutes a need for the second event to occur.
- A fuzzy system is defined as a system with operating principles based on fuzzy information processing and decision making.
- Designing a fuzzy system involves developing mechanisms for fuzzy information processing and decision making within digital platform and soft computing environment.

3.6 Fuzzy Rule Based System

- **The fuzzy algorithm is an organized collection of the theoretical elements implemented in a computer program.**

- **It is a procedure for performing a task formulated as a collection of a fuzzy IF-THEN rules.**

- **The input-output relationship or rules are easily expressed using IF-THEN statements such as the following:**

IF x is A_1 THEN y is B_1 ELSE

IF x is A_2 THEN y is B_2 ELSE

.

.

IF x is A_n THEN y is B_n

3.6 Fuzzy Rule Based System

- **There are three basic domains of information in a fuzzy algorithm**

    - **input data**
    - **output data**
    - **design data**

- **Input data and output data are dynamically updated, whereas design data are permanently stored in memory.**

- **Design data embody the knowledge of a specific solution.**

- **Each rule is represented by an implication relation R(x,y) and the form of R(x,y) depends on the implication operator.**

3.6 Fuzzy Rule Based System

➢ **Following table shows the most common interpretations of the connective ELSE for implication operators.**

Implication	Interpretation of ELSE
Zadeh max-min $Ø_m$	AND (∧)
Mamdani min $Ø_c$	OR (∨)
Larsen product $Ø_p$	OR (∨)
Arithmetic $Ø_a$	AND (∧)
Boolean $Ø_b$	AND (∧)
Bounded product $Ø_{BP}$	OR (∨)
Drastic product $Ø_{DP}$	OR (∨)
Standard sequence $Ø_s$	AND (∧)
Gougen Ø	AND (∧)
Godelian $Ø_g$	AND (∧)

3.6 Fuzzy Rule Based System

➢ **A relation encompassing the entire collection of rules is known as an *algorithmic relation*.**

$$R_{\sim}(x, y) = \int_{(x,y)} «_{\sim}(x, y)/(x, y) \quad ....(3.35)$$

➢ **The operator AND is either the union (∨) or the intersection (∧) of the implication relations of the individual rules.**

➢ **Given a new fuzzy value A', we evaluate it using GMP,**

IF x is A_1 THEN y is B_1 ELSE

IF x is A_2 THEN y is B_2 ELSE

.

.

IF x is A_n THEN y is B_n

x is A'

---

y is B'

3.6 Fuzzy Rule Based System

- The output value **B'** is computed by the max-min composition of **A'** and **R_α(x,y)**.

$$B' = A' \circ \underset{\sim}{R}(x, y) \quad ....(3.36)$$

- The membership function of **B'** is

$$\mu_{B'}(y) = \bigvee_{x} [\mu_{A'}(x) \wedge \mu_{\underset{\sim}{R}}(x, y)] ....(3.37)$$

- The inverse problem is solved using **GMT**

IF x is A_1 THEN y is B_1 ELSE

IF x is A_2 THEN y is B_2 ELSE

:

IF x is A_n THEN y is B_n

y is B'

---

x is A'

3.6 Fuzzy Rule Based System

- The membership function of **A'** can be computed by the max-min composition of **R_α(x,y)** and **B'**.

$$A' = \underset{\sim}{R}(x, y) \circ B' \quad ....(3.38)$$

- The membership function of **A'** is

$$\mu_{A'}(x) = \bigvee_{y} [\mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{B'}(y)] ....(3.39)$$

3.7 Defuzzification Methods

- Aggregating two or more fuzzy output sets yields a new fuzzy set in the basic fuzzy inference algorithm.

- In most cases, a result in the form of a fuzzy set is converted into a crisp result by the **defuzzification** process.

- Selecting a crisp number **x** representative of **μ(x)** is a process known as **defuzzification**.

- Following **defuzzification** techniques are used frequently

  - Centre of Area
  - Centre of Sum
  - Mean of Maxima

3.7 Defuzzification Methods

3.7.1 Centre of Area Defuzzification

- In **centre of area** defuzzification, the crisp value **x** is taken to be the geometrical centre of the output fuzzy value **μ(x)**.

- The **μ(x)** is formed by taking the union of the contributions of all the rules whose **DOF** is greater than zero.

- The centroid methods are based on finding the balance point of the whole geometric figure or the area of each fuzzy set.

- Let us assume we have a discrete universe of discourse.

- The **defuzzification** output is defined as

$$x = \frac{\sum_{i=1}^{n} x_i . \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)} ......(3.40)$$

3.7 Defuzzification Methods

3.7.1 Centre of Area (COA) Defuzzification

- **Where the summation is carried out over discrete values of the universe of discourse, μ_i, sampled at n points.**
- **COA takes into account the area of the resultant membership function.**
- **If the areas of two or more contributing rules overlap, COA does not take into account the overlapping area.**
- **The approximation embedded in above equation will be justified when the membership function is defined as a fuzzy set comprising singletons.**
- **The membership function $\mu(x)$ represents the fuzzy set of the final output of one linguistic or fuzzy variable, and x is the location of a singleton in the universe of discourse.**

3.7 Defuzzification Methods

3.7.2 Centre of Sum Defuzzification

- **In COA method, the overlapping area is counted once.**
- **To address the problems associated with COA and to take into account the overlapping areas of multiple rules more than once, a variant of COA called centre of sums (COS) is used.**
- **In case of COS, the overlapping areas are counted more than once, and hence the COS defuzzification method is most commonly used.**
- **The defuzzification output is defined as**

$$x = \frac{\sum_{i=1}^{N} x_i \sum_{k=1}^{n} \mu_{A_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{A_k}(x_i)} \quad \text{......(3.41)}$$

- **N -- No. of sample points**
- **n -- No. of rules**

3.7 Defuzzification Methods

3.7.3 Mean of Maxima

- **One simple way of defuzzifying output is to take the crisp value with the highest degree of membership.**

- **In cases with more than one element having the maximum value, the mean value of the maxima is taken.**

$$x = \frac{\sum_{x_i \in M} x_i}{|M|} \quad ......(3.42)$$

Where M = {x_i|$\mu(x_i)$ is equal to the height of fuzzy set}

Reference:

1) Neural Networks, Fuzzy Logic, and Genetic Algorithms: Synthesis and Applications by S. Rajasekaran, G.A.V. Pai. PHI.

2) Artificial Intelligence and Intelligent Systems by N P Padhy, Oxford Press.